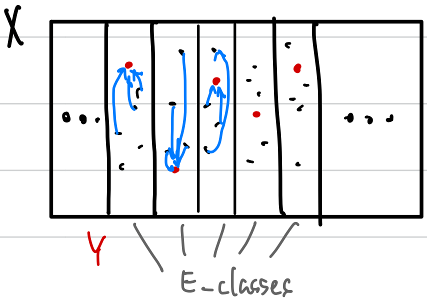
Math 564: Real analysis and measure theory
Lecture

Nonmeasurable sets.

We will give an example of a non-Lebesgue-measurable subset of $\mathbb{R}$.

Def. Let E be an equivalence relation on a set X. A *transversal* for E is a set $Y \subseteq X$ which meets each E-class in exactly one point. A *selector* for E is a map $s : X \to X$ such that $s(x) \in [x]_E$ and $x E y \Leftrightarrow s(x) = s(y)$ for all $x, y \in X$. For a selector s, we can get a transversal $Y := s(X)$, and vice versa, from a transversal Y, we get a selector by $s(x) :=$ the unique $y \in Y \cap [x]_E$.



Selectors and transversals exist by Axiom of Choice, but this typically results in ill-behaved functions and sets, for example, nonmeasurable.

Example. Let $E_{\mathbb{Q}}$ be the so-called *Vitali equivalence relation* on $\mathbb{R}$ defined by

$$x E_{\mathbb{Q}} y \; :\Leftrightarrow \; y - x \in \mathbb{Q}.$$

This is simply the coset equivalence relation of $\mathbb{Q}$ as a subgroup of $\mathbb{R}$ under addition. Also, this is the orbit equivalence relation of the action of $\mathbb{Q}$ on $\mathbb{R}$ by translation. For each $x \in \mathbb{R}$, the class $[x]_{E_{\mathbb{Q}}} = x + \mathbb{Q}$, in particular, it intersects $[0,1]$.

Claim. Any transversal $Y \subseteq [0,1]$ of $E_{\mathbb{Q}}$ is nonmeasurable wrt Lebesgue measure λ.

Proof. Just observe that

$$[0,1] \subseteq \bigsqcup_{q \in \mathbb{Q} \cap [-1,1]} q + Y \subseteq [-1,2].$$

If Y was measurable, so would be its translates $q + Y$ and $\lambda(q+Y) = \lambda(Y)$. So:

$$1 = \lambda([0,1]) \leq \lambda\Big(\bigsqcup_q q + Y\Big) = \sum_q \lambda(q+Y) = \infty \cdot \lambda(Y) \leq \lambda([-1,2]) = 3,$$ a contradiction. $\square$

Remark. It is tempting to think that nonmeasurable sets can only arise from Axiom of Choice. This has some truth to it but not entirely. Indeed, in Solovay's model of ZF (Zermelo-Fraenkel set theory without Choice) where Axiom of Choice fails badly, all subsets of $\mathbb{R}$ are Lebesgue measurable. On the other hand, there are simple constructions of subsets of $\mathbb{R}$ without Choice, which yield non-measurable sets in some other models of ZF. More concretely, there is a G_δ (ctbl intersection of open sets) subset B of $\mathbb{R}^3$, such that whether or not the set $\mathrm{proj}_{\mathbb{R}}(\mathbb{R}^2 \setminus \mathrm{proj}_{\mathbb{R}^2}(B))$ is independent of ZFC!!! This is to say that measurability is a subtle property, and even the measurability of projections of Borel sets (called analytic sets) is a difficult theorem.

Pocket tools for working with measures.

Prop (monotone convergence). Let $(X, \mathcal{B}, \mu)$ be a measure space.

(a) $\mu\left(\bigcup\limits_{n\in\mathbb{N}}^{\uparrow} A_n\right) = \lim\limits_{n\to\infty} \mu(A_n)$ for all μ-measurable A_n with $A_n \subseteq A_{n+1}$.

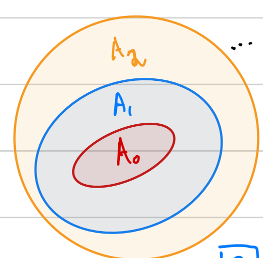
(b) $\mu\left(\bigcap\limits_{n\in\mathbb{N}}^{\downarrow} A_n\right) = \lim\limits_{n\to\infty} \mu(A_n)$ for all μ-meas. B_n with $B_n \supseteq B_{n+1}$ and $\mu(B_0) < \infty$.

Caution. If all B_n have infinite measure, part (b) may not hold: take $B_n := (n, \infty)$, then $\bigcap\limits_{n\in\mathbb{N}}^{\downarrow} B_n = \emptyset$ so it is null, but $\lim\limits_{n\to\infty} \lambda(B_n) = \infty$.

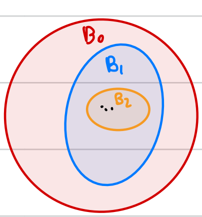
$A_0' := A_0$

Proof. (a) We disjointify: $A_n' := A_n \setminus A_{n-1}$, so $\bigsqcup\limits_{n\in\mathbb{N}} A_n' = \bigcup\limits_{n\in\mathbb{N}}^{\uparrow} A_n$, hence

$$\mu\left(\bigcup_{n\in\mathbb{N}}^{\uparrow} A_n\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} A_n'\right) = \sum_{n\in\mathbb{N}} \mu(A_n')$$



$$= \lim_{N\to\infty} \sum_{n\le N} \mu(A_n') = \lim_{N\to\infty} \mu\left(\bigsqcup_{n<N} A_n'\right) = \lim_{N\to\infty} \mu(A_N). \quad \boxed{a}$$



(b) The sets $A_n := B_0 \setminus B_n$ are increasing, so by (a), we have

$$\mu\left(\bigcup_{n}^{\uparrow} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \left(\mu(B_0) - \mu(B_n)\right) = \mu(B_0) - \lim_{n\to\infty} \mu(B_n),$$

where we use that $\mu(B_n) \le \mu(B_0) < \infty$ in the second equality. On the other hand:

$$\mu\Big(\bigcup_{n\in\mathbb{N}}^{\uparrow} A_n\Big) = \mu\Big(B_0 \setminus \bigcap_{n\in\mathbb{N}}^{\downarrow} B_n\Big) \overset{\mu(\bigcap_n B_n)<\infty}{=} \mu(B_0) - \mu\Big(\bigcap_{n\in\mathbb{N}}^{\downarrow} B_n\Big), \text{ so } \mu\Big(\bigcap_{n\in\mathbb{N}}^{\downarrow} B_n\Big) = \lim_{n\to\infty} \mu(B_n). \qquad \square$$

Borel-Cantelli lemmas (Important Pigeonhole Principles). Let $(X,\mathcal{B},\mu)$ be a measure space. Let (A_n) be a sequence of μ-measurable sets.

(a) If $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then a.e. $x\in X$ is eventually not in A_n, i.e. the set

$$\limsup_{n\to\infty} A_n := \{x\in X : \underbrace{\exists^\infty_n}_{\forall m\, \exists n\geq m} \; x \in A_n\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n$$

is μ-null.

(b) (Measure compactness). Suppose $\mu(X)<\infty$. If $\exists \delta>0$ s.t. $\mu(A_n)\geq\delta$ for all $n\in\mathbb{N}$ then $\mu(\limsup_{n\to\infty} A_n) \geq \delta$.

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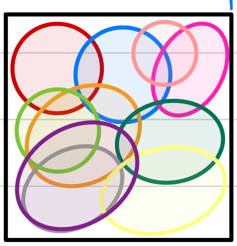

Proof. (a) Note that $\limsup_n A_n \subseteq \bigcup_{n\geq m} A_n$ for all $m\in\mathbb{N}$, so

$$\mu(\limsup_n A_n) \leq \mu\Big(\bigcup_{n\geq m} A_n\Big) \leq \sum_{n\geq m} \mu(A_n) \longrightarrow 0 \text{ as } m\to\infty.$$

(b) Since $\mu(X)<\infty$, $\mu(\limsup_n A_n) = \mu\Big(\bigcap_m^{\downarrow} \bigcup_{n\geq m} A_n\Big) \overset{\text{previous Prop (b)}}{=} \lim_{m\to\infty} \mu\Big(\bigcup_{n\geq m} A_n\Big) \geq \delta.$ $\square$

Application. Let $(X,\mathcal{B},\mu)$ be a measure space. A sequence (V_n) of μ-meas. sets is called *vanishing* (resp. *almost vanishing*) if (V_n) is decreasing and $\bigcap_{n\in\mathbb{N}}^{\downarrow} V_n$ is empty (resp. null).

Prop. Let $\mathcal{F}$ be a collection of μ-meas sets that is closed under ctbl unions. If $\mathcal{F}$ contains positive measure sets of arbitrarily small measure, then $\mathcal{F}$ contains an almost vanishing sequence of positive measure sets.

Proof. For each $n\in\mathbb{N}$, let $A_n\in\mathcal{F}$ be a positive meas. set with $\mu(A_n)\leq 2^{-n}$. The sets A_n may not be decreasing, but the sets $V_n := \bigcup_{m\geq n} A_m$ are decreasing and $\bigcap_{n\in\mathbb{N}}^{\downarrow} V_n = \limsup_{n\to\infty} A_n$ is null by Borel-Cantelli because $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$. $\square$